MathsCraft

Doing maths like a research mathematician

Key concepts

Listed here are some key concepts underlying the MathsCraft approach. The sections are organised according to some common themes, but can be read independently.

CONTENTS

HOW TO HAVE IDEAS

TIME-LAG

We believe that a person must have had sufficient time to become skilled at using a (mathematical) tool in order to be able to wield it well while attempting to solve a problem. Partly this has to do with experience with the tool and knowing how to use it, and partly this has simply to do with how easily it will occur to you to use it.

With a group of students

Of course this implies that someone trying to solve a problem may need to already be skilled with a wide range of tools. This is why the MathsCraft Curriculum is *supplementary* to the standard curriculum – the standard curriculum is the place where students gain those tools.

As a rule of thumb we consider a problem appropriate for students when the tools they need to access it are ones they learnt (in normal maths classes) around 3 years ago. Hence some MathsCraft problems may appear on the surface to be quite simple for their intended audience.

But rest assured: there is plenty of beautiful, interesting, challenging mathematics that can be done with very simple tools.

DATA

When starting out on a problem, it can be useful to have a number of examples to look at. We call these data, using the term broadly. Generating data can allow you to get a feel for the structure of the problem, and through looking at the data you can notice things that may give insight. Not everything you notice will be significant, but the more you look the more you will see.

With a group of students

There's sometimes an opportunity to have each person generate a small amount of data, and then pool it by having a number of them write what they've got on the board. Data on the board is available for everyone to look at, so everyone has the opportunity to notice things.

Once conjectures are made that agree with all the available data, more data can be generated to test the conjecture. But keep in mind that even if the conjecture is supported by the new data, it still may be false (see BAD arguments).

OBSERVATIONS

To get to the bottom of a problem, a way in can be to simply look at what you've got so far (usually what you've got is Data) and see what you can observe. Observations are usually the starting point when trying to solve a problem – but we can't control what we observe! So the best approach is to simply observe anything you can. At this point, no fruit is too low — anything might be relevant. Whatever turns out not to be relevant can be put on The Shelf, or just put aside.

You can start by just trying to spot any patterns, but it can also be worthwhile to really try to think about what's going on: what the underlying rules are and how they might be producing the patterns you're seeing. This takes more mental effort but can be more fruitful.

Observations can themselves act as a form of data. Having a list of your observations written down in front of you, so you can look at them, may lead you to see connections between them.

Observation can also play a big role earlier in the process, it can be a way to find a problem to work on. A simple example is looking at a mathematical object, observing a property or two, and asking "how many other objects have those properties, and what else do they have in common?".

With a group of students

As with data, there may be an opportunity for students to collect observations, and then share them with the room. Writing these on the board yourself will allow you to keep control of how clearly they're written, and avoid double-ups. Once written on the board they can provide fuel for more observations, or spark ideas or questions.

CONVERSATIONS AND DECISIONS

DEFINITIONS

In maths you will come across a lot of definitions. Each one has been decided on by someone, for a reason. And each one could have been defined differently.

Definitions are about communication: it's about making things precise so that when you talk to someone else you each know exactly what you're talking about.

But it's also about communicating with yourself. Definitions help you refer to a concept in your head, or on paper, so that you can think about it more easily (see also Notation).

Some definitions are set by convention — meaning that the mathematical community has broadly agreed on what that thing means. Following convention for such things makes communicating with others easier. But that doesn't mean those definitions are "correct", any more than the word "rock" is the correct word for a rock.

A lot of the time, the definition of any word, symbol or thing is up to you — choose a definition that seems like it will be most useful: one that makes things clear and easy to think about.

If things are hazy, you keep getting confused, or you're arguing too much with the people you're working with, it could be a sign that you need to go back and carefully define something.

With a group of students

Once far enough into the problem, there will often be arguments about what things mean. If there's disagreement on what the most useful definition will be, a simple way to avoid endless debate is to put it to a vote. If it later turns out it would have been better to define things differently, you can always go back and change your definition!

RULES

There are a lot of rules in maths. But these rules weren't handed down from on high, and a lot of new maths is invented by asking "what would happen if I broke this rule?". Some of the simplest examples lead to very deep questions: For example, you can't divide by 0. But let's say you can — what would 1/0 mean? How is it related to infinity? Does arithmetic apply to infinity? If not, can we change the rules of arithmetic to make it apply?

Each piece of mathematics is based on a set of assumptions, definitions, and rules. But these can be changed — what's really important is to remain internally consistent. You can create a thing, make it follow some rules, and then find out what the consequences of those rules are.

So when faced with a set of rules in a problem, you can choose to stick to them, or you can choose to change them (see Think before - Think within - Think beyond). As long as you know clearly what rules you are following, it's all good!

With a group of students

Students seem to be used to the idea that they will be told what the rules are, and then they have to follow them.

Something fun to do (and often a new experience) is to let the students make the rules. This can be done by "starting loose" — being ambiguous about the rules. It can help to be not-obviously-ambiguous, because this will mean the students get to work and make assumptions (without realising it), rather than asking you to clarify. Then you can use the Data or Observations stages to expose these assumptions, have the students argue over the rules or definitions, and insist that the room come to an agreement, one way or another.

NOTATION

Notation can be a blessing and a curse. Firstly, standard notation for things is used for communication (for more on this see Definitions). But much like definitions, if you don't need to communicate with anyone else yet, you can invent your own notation.

Why would you want to do this, if notation already exists? Well, different notation can help you think about things differently. Sometimes, one aspect of a thing is not as important as another. You can choose or invent a notation that will hide the aspects you don't care about, and make obvious the ones you do. For example, if all that matters about a number is whether it's even or odd, consider a notation which masks the actual number but highlights evenness or oddness: represent numbers by 1/0, O/E, or so on.

You may need to change notation if the important aspects change.

With a group of students

Invention is a huge part of doing maths. Students will invent all sorts of notation. As with definitions, there's usually no right or wrong, just pros and cons to each choice of notation. Having different students use the same notation can be helpful to them when sharing things, but is not necessary – things can be learnt from seeing someone else's way of doing things.

If a student uses a notation that you think will be helpful to the rest of the class, you can have them share it and point out the benefits of the notation. Students will naturally take it up if they think it will be useful to them.

BAD ARGUMENTS

ARE YOU SURE, SURE *BAD*?

There's being sure, and then there's being sure BAD — "Beyond All Doubt". This is MathsCraft slang for a proof: an argument that begins with assumptions and follows logical steps to reach a conclusion. If you have a BAD argument for something, you know that if the assumptions are true the conclusion must be true.

The trick is finding the argument and knowing that it's a BAD one. We can easily convince ourselves that something is true when there's actually a hole in the argument – something we've overlooked or an error in logic. It's a good idea to practice scrutinizing your own argument and looking for flaws, but don't just rely on yourself: Often the best way to find flaws is to get someone else to scrutinize it.

With a group of students

The first step in allowing students to come up with BAD arguments is to not give them answers, and to not tell them if they're right or wrong. If they have an answer and they want to know if it's right, they should convince themselves and/or those they're working with. (In the world of maths research, there are no answers at the back of the book.)

Students' arguments will often be full of holes - you can get good at spotting them and deciding how to respond. It's good if students can learn to spot the holes themselves, but sometimes they will need a pointer: something like "that's a nice argument, but I'm not quite convinced by this bit."

THE BOOK

The (in-)famous mathematician Paul Erdős was known for referring to what he simply called "The Book" — the book in which God keeps the most elegant proofs of mathematical theorems. He once said "You don't have to believe in God, but you should believe in The Book."

Theorems can often be proved in multiple ways. Sometimes the proof is simply logically sound and correct (a BAD argument), which means the theorem is true. But sometimes the proof is also illuminating: It shows not only that the theorem is true but also why it's true. It lays bare the structure underlying the fact. These proofs are said to be more elegant. Usually the simpler the better.

There is usually little doubt when a proof is elegant. There is something aesthetically pleasing about it. When you create an elegant proof, you know it.

You can read more about The Book on Wikipedia.

EXPLORATION

THINK BEFORE - THINK WITHIN - THINK BEYOND

"Think beyond" is a phrase we use to suggest you try breaking the Rules, after having found a solution to the current problem. Often an easy way to do this is to change a number or parameter: if you've been working with a size 4 something-or-other, why not try size 5? Size 6? Size N?

This can be the same thing as "generalising". Thinking beyond, considering problems that are related but not quite the same, can show you how the one you've already considered fits into a bigger picture, and can often turn up new unexpected maths.

"Think before" is similar but in the other direction: to use our earlier example, what about size 3? Size 2? Size 1? The small cases might not seem worth bothering with, but they can show how the rules work at the most basic level, which can give you new insights into the problem. And if the example is really simple, they won't take much of your time!

"Think within" is about taking another look at what you've done, and looking for another perspective, or connections that maybe you didn't see before.

With a group of students

The key to these concepts is timing. Students who are not ready to move on from their current train of thought, won't.

THE SHELF

There are two general sorts of modes you can be in when doing maths — one is looking for a problem to solve, the other is trying to solve a problem. No matter which mode you're in, you're bound to have ideas about potential tangents. These can be ways to break the rules you're working with, or things you notice, or just a question you think of while working (see Rules, Observations, and Think before - Think within - Think beyond).

The more you're into a problem the more this will happen — you need to have played around a bit in order to think of things. But you may be in the second mode, trying to focus on achieving a task. This is where "the shelf" comes in.

Rather than getting distracted by the tangent, or alternatively saying "that doesn't follow the rules" or "that's irrelevant" and forgetting about it, you can put the idea *on the shelf* and maybe take it down later to investigate. It might lead nowhere or it might lead somewhere really interesting, and there's only one way to find out.

By the way, this is how mathematicians find new problems to work on.

With a group of students

You could keep a list in the corner of the board of things that have come up that you don't want the room to be thinking about right now — visibly "put it on the shelf" so that the students know it hasn't been dismissed, and they can think about it later if they want.

You could also suggest students keep a list of these kinds of ideas in the back of their book.